Proof By Induction Notes

Introduction:

- In a proof by induction, there are 3 main parts:

  1. Base Case
  2. Hypothesis Step
  3. Induction Step

  Note: Sometimes, these 2 steps get merged into 1.

- In the base case, you want to show that the formula works/holds for the lowest possible value.

- In the hypothesis step, you want to assume that the formula holds for an arbitrary value, k.

- In the induction step, you want to prove that the formula holds for k+1.

Examples:

1. Using induction, prove that the following formula is true for all $n \in \mathbb{Z}^+$:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Solution:

Base Case $(n=1)$:

$LHS = 1$

$$RHS = \frac{(1)(2)}{2} = 1$$

$LHS = RHS$

$\therefore$ The formula holds for the base case.

Hypothesis Step:
Assume that the formula holds for any
$k \geq 1,\ k \in \mathbb{Z}^+$

Induction Step:
Want to prove (WTP) that the formula holds
for $k+1$

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$$

$$= \underbrace{\frac{k(k+1)}{2}}_{\text{By Hypothesis Step}} + k+1$$

$$= \frac{k^2+k+2k+2}{2}$$

$$= \frac{k^2+3k+2}{2}$$

$$= \frac{(k+2)(k+1)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2}$$

$\therefore$ By proof of induction, the formula holds for
all $n \in \mathbb{Z}^+$

2. Using induction, prove that $S(n)$ holds for all $n \in \mathbb{Z}^+$

$S(n)$: $5 + 10 + 15 + \ldots + 5n = \frac{5n(n+1)}{2}$

Soln:

Base Case $(n=1)$:

Let $n = 1$

$LHS = 5$

$RHS = \frac{5(2)}{2}$

$= 5$

$LHS = RHS$

$\therefore$ $S(n)$ holds for the base case.

Hypothesis Step:

Assume that $S(n)$ holds for $n=k$, $k \geq 1$, $k \in \mathbb{Z}^+$

Induction Step:

WTP: $S(n)$ holds for $k+1$

$$5 + 10 + \ldots + 5(k+1) = 5 + \ldots + 5(k) + 5(k+1)$$

$$= \underbrace{\frac{5k(k+1)}{2}}_{\text{By Hypothesis Step}} + 5(k+1)$$

$$= \frac{5k^2 + 5k + 10k + 10}{2}$$

$$= \frac{5(k^2 + 3k + 2)}{2}$$

$$= \frac{5((k+1)(k+2))}{2}$$

$$= \frac{5((k+1)((k+1)+1))}{2}$$

$\therefore$ By induction, we've proven that $S(n)$ holds for all $n \in \mathbb{Z}^+$.

3. Using induction, prove that $S(n)$ holds for all $n \in \mathbb{N}$

$S(n)$: $n(n^2+5)$ is divisible by 6

Solution:

Base Case $(n=0)$:
Let $n=0$
$0(0^2+5)=0$, which is divisible by 6
$\therefore$ $S(n)$ holds for the base case.

Hypothesis Step:
Assume $S(n)$ holds for some $n=k$, where $k \in \mathbb{N}$.

Induction Step:
WTP: $S(n)$ holds for $k+1$.

$$\begin{aligned}(k+1)((k+1)^2+5) &= (k+1)(k^2+2k+1+5) \\ &= k(k^2+2k+1+5)+(k^2+2k+1+5) \\ &= \underline{k^3}+\underline{2k^2}+\underline{k}+\underline{5k}+\underline{k^2}+\underline{2k}+\underline{6} \\ &= \underline{k(k^2+5)}+\underline{(3k^2+3k)}+\underline{6} \\ &= k(k^2+5)+3k(k+1)+6\end{aligned}$$

We know from the hypothesis step that $k(k^2+5)$ is divisible by 5. $k(k^2+5) = k^3 + 5k$. Hence, we know that the first term is divisible by 6.

For the second term, $3k(k+1)$, either k or $k+1$ must be even. $3 \cdot$ (even num) is divisible by 6. Hence, the second term is also divisible by 6.

6 is obviously divisible by 6.

Hence, the entire RHS is divisible by 6.

$\therefore$ $S(n)$ holds for all $n \in \mathbb{N}$.